ASYMPTOTIC DIRECTIONS IN RANDOM WALKS IN RANDOM ENVIRONMENT REVISITED

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Abstract

Recently Simenhaus in [Sim07] proved that for any elliptic random walk in random environment, transience in the neighborhood of a given direction is equivalent to the a.s. existence of a deterministic asymptotic direction and to transience in any direction in the open half space defined by this asymptotic direction. Here we prove an improved version of this result and review some open problems.

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1 Introduction

For each site $x \in \mathbb{Z}^d$, consider the vector $\omega(x) := \{\omega(x,e) : e \in \mathbb{Z}^d, |e| = 1\}$ such that $\omega(x,e) \in (0,1)$ and $\sum_{|e|=1} \omega(x,e) = 1$. We call the set of possible values of these vectors $\mathcal{P}$ and define the *environment* $\omega = \{\omega(x) : x \in \mathbb{Z}^d\} \in \Omega := \mathcal{P}^{\mathbb{Z}^d}$. We define a random walk on the random environment ω, as a random walk $\{X_n : n \in \mathbb{N}\}$ with a transition probability from a site $x \in \mathbb{Z}^d$ to a nearest neighbor site $x + e$ with $|e| = 1$ given by $\omega(x,e)$. Let us call $P_{x,\omega}$ the law of this random walk starting from site x in the environment ω. Let $\mathbb{P}$ be a probability measure on Ω such that the coordinates $\{\omega(x)\}$ of ω are i.i.d. We call $P_{x,\omega}$ the *quenched* law of the random walk in random environment (RWRE), starting from site x. Furthermore, we define the *averaged* (or *annealed*) law of the RWRE starting from x by $P_x := \int_\Omega P_{x,\omega}\, d\mathbb{P}$. In this note we discuss some aspects of RWRE related to the a.s. existence of an asymptotic direction in dimension $d \geq 2$, briefly reviewing some of the open questions which have been unsolved and proving an improved version of a recent theorem of Simenhaus on the a.s. existence of an asymptotic direction.

Some very fundamental and natural questions about this model remain open. Given a vector $l \in \mathbb{R}^d \backslash \{0\}$, define the event

$$A_l := \{\lim_{n\to\infty} X_n \cdot l = \infty\}.$$

Whenever A_l occurs, we say that the random walk is transient in the direction l. Let also

$$B_l := \left\{ \liminf_{n\to\infty} \frac{X_n \cdot l}{n} > 0 \right\}.$$

Whenever B_l occurs, we say that the random walk is ballistic in the direction l. We have the following open problem.

Open problem 1.1. *In dimensions $d \geq 2$, transience in the direction l implies ballisticity in the direction l.*

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Some partial progress related to this question has been achieved by Sznitman and Zerner [SZ99], and later by Sznitman in [Szn00, Szn01, Szn02], which we will discuss below. Under a uniform ellipticity assumption, i.e. $\mathbb{P}(\operatorname{ess\,inf}\min_{|e|}\omega(0,e) > 0) = 1$, the following lemma, which we call Kalikow's zero-one law, was proved by Sznitman and Zerner (cf. Lemma 1 in [SZ99]) using regeneration times. Later Zerner and Merkl [ZM01] derived the corresponding result under the assumption of ellipticity only, i.e. $\mathbb{P}(\min_{|e|}\omega(0,e) > 0) = 1$; cf. Proposition 3 in [ZM01].

Lemma 1.2 (Sznitman-Zerner). *For $l \in \mathbb{R}^d \backslash \{0\}$,*

$$P_0(A_l \cup A_{-l}) = 0 \quad \text{or} \quad 1.$$

On the other hand, in dimension $d = 1$ a zero-one law holds, i.e. $P_0(A_l) \in \{0,1\}$. Zerner and Merkl, proved the following (see Theorem 1 in [ZM01]).

Theorem 1.3 (Zerner-Merkl). *In dimension $d = 2$, for $l \in \mathbb{R}^2 \backslash \{0\}$,*

$$P_0(A_l) = 0 \quad \text{or} \quad 1.$$

Nevertheless, we still have the following open problem.

Open problem 1.4. *In dimensions $d \geq 3$, for $l \in \mathbb{R}^d \backslash \{0\}$,*

$$P_0(A_l) = 0 \quad \text{or} \quad 1.$$

Combining Kalikow's zero-one law with the law of large numbers result of Sznitman and Zerner in [SZ99], Zerner [Zer02] proved the following theorem.

Theorem 1.5 (Sznitman-Zerner). *In dimensions $d \geq 2$, there exists a direction $\nu \in \mathbb{S}^{d-1}$, $\nu \neq 0$, and $v_1, v_2 \in [0,1]$ such that P_0-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{n} = v_1 \nu \mathbb{1}_{A_\nu} - v_2 \nu \mathbb{1}_{A_{-\nu}}.$$

Indeed, Theorem 3.2.2 of [TZ04], the proof of which can be performed in the same manner with the assumption of ellipticity only instead of uniform ellipticity, implies that for $e \in \mathbb{Z}^d$ with $|e| = 1$ and

$$P_0(A_e \cup A_{-e}) = 1 \tag{1.1}$$

there exist $v_e, v_{-e} \in [0,1]$ such that P_0-a.s.

$$\lim_{n\to\infty} \frac{X_n e}{n} = v_e \mathbb{1}_{A_e} - v_{-e} \mathbb{1}_{A_{-e}}. \tag{1.2}$$

Combining this with Theorem 1 of [Zer02] we may omit assumption (1.1) and still obtain (1.2). Having (1.2) for the elements $e_1, \ldots, e_d$ of the standard basis of $\mathbb{R}^d$, we obtain that $\lim_{n\to\infty} X_n/n$ exists P_0-a.s. and may take values in a set of cardinality 2^d. Employing the same argument as Goergen in p. 1112 of [Goe06] we now obtain that P_0-a.s. $\lim_{n\to\infty} X_n/n$ takes two values at most. This yields Theorem 1.5.

Whenever $\lim_{n\to\infty} X_n/|X_n|$ exists P_0-a.s. we call this limit the *asymptotic direction* and we say that a.s. an asymptotic direction exists. The existence of an asymptotic direction can already be established assuming some of the conditions introduced by Sznitman which imply ballisticity. Let $\gamma \in (0,1)$ and $l \in \mathbb{S}^{d-1}$. The condition $(T)_\gamma$ holds relative to l if for all $l' \in \mathbb{S}^{d-1}$ in a neighborhood of l,

$$\limsup_{L\to\infty} L^{-\gamma} \log P_0(\{X_{T_{U_{l',b,L}}} \cdot l' < 0\}) < 0, \tag{1.3}$$

for all $b > 0$, where $U_{l',b,L} = \{x \in \mathbb{Z}^d : -bL < x \cdot l' < L\}$ is a slab and $T_{U_{l',b,L}} = \inf\{n \geq 0 : X_n \notin U_{l',b,L}\}$ is the first exit time of this slab. On the other hand, one says that condition (T') holds relative to l if condition $(T)_\gamma$ holds relative to l for every $\gamma \in (0,1)$. It is known that for each $\gamma \in (0,1)$ condition $(T)_\gamma$ relative to l implies transience in the direction l and that a.s. an asymptotic direction exists which is deterministic. Also, for each $\gamma \in (1/2,1)$, condition $(T)_\gamma$ relative to l implies condition (T'), which in turn implies ballisticity (see [Szn02]). One of the open problems related to condition $(T)_\gamma$ is the following.

Open problem 1.6. *If (1.3) is satisfied for $l' \in \mathbb{S}^{d-1}$, then $(T)_\gamma$ holds relative to l'.*

Recently in [Sim07], Simenhaus established the following theorem which gives equivalent conditions for the existence of an a.s. asymptotic direction and showing that transience in a neighborhood of a given direction implies that a.s. an asymptotic direction exists.

Theorem 1.7 (Simenhaus)**.** *The following are equivalent:*

(a) There exists a non-empty open set $O \subset \mathbb{R}^d$ such that

$$P_0(A_l) = 1 \quad \forall\, l \in O. \tag{1.4}$$

(b) There exists $\nu \in \mathbb{S}^{d-1}$ such that P_0-a.s.

$$\lim_{n\to\infty} \frac{X_n}{|X_n|} = \nu.$$

(c) There exists $\nu \in \mathbb{S}^{d-1}$ such that $P_0(A_l) = 1$ for all $l \in \mathbb{R}^d$ with $l \cdot \nu > 0$.

It is natural to wonder if there exists a statement analogous to Theorem 1.5, but related only to the existence of a possibly non-deterministic asymptotic direction. Here we answer affirmatively this question proving the following generalization of Theorem 1.7.

Theorem 1.8. *The following are equivalent:*

(a) There exists a non-empty open set $O \subset \mathbb{R}^d$ such that

$$P_0(A_l \cup A_{-l}) = 1 \quad \forall\, l \in O.$$

(b) There exist d linearly independent vectors $l_1, \dots, l_d \in \mathbb{R}^d$ such that

$$P_0(A_{l_k} \cup A_{-l_k}) = 1 \quad \forall\, k \in \{1, \dots, d\}. \tag{1.5}$$

(c) There exists $\nu \in \mathbb{S}^{d-1}$ with $P_0(A_\nu \cup A_{-\nu}) = 1$ such that P_0-a.s.

$$\lim_{n\to\infty} \frac{X_n}{|X_n|} = \mathbb{1}_{A_\nu}\nu - \mathbb{1}_{A_{-\nu}}\nu. \tag{1.6}$$

(d) There exists $\nu \in \mathbb{S}^{d-1}$ such that

$$P_0(A_l \cup A_{-l}) = 1$$

if and only if $l \in \mathbb{R}^d$ is such that $l \cdot \nu \neq 0$. In this case, $P_0(A_l \Delta A_\nu) = 0$ and $P_0(A_{-l} \Delta A_{-\nu}) = 0$ for all l such that $l \cdot \nu > 0$.

It should be noted that Theorems 1.7 and 1.8 are interesting only in the case in which the statement of the Open Problem 1.1 is not proven to be true. Furthermore, if condition (1.5) is fulfilled but (1.4) is not, then if asymptotic directions exist we have to expect at least (and as it turns out at most, see also Proposition 1 in [Sim07]) two of them. However, it is not known whether condition (1.5) can be fulfilled while (1.4) is not. In fact, if the statement of the Open Problem 1.4 holds, then the two conditions are equivalent. Note that due to Kalikow's zero-one law, condition (d) of Theorem 1.8 yields a complete characterisation of $P_0(A_l \cup A_{-l})$ for all $l \in \mathbb{R}^d$. As a consequence of this result, we obtain an a priori sharper version of (c) in Theorem 1.7:

(c') *There exists $\nu \in \mathbb{S}^{d-1}$ such that $P_0(A_l) = 1$ for all $l \in \mathbb{R}^d$ with $l \cdot \nu > 0$ and $P_0(A_l) = 0$ if $l \cdot \nu \leq 0$.*

This observation and Theorem 1.3 imply that in dimension $d = 2$ there are at most three possibilites for the values of the set of probabilities $\{P_0(A_l) : l \in \mathbb{S}^{d-1}\}$: (1) for all l, $P_0(A_l) = 0$; (2) there exists a $\nu \in \mathbb{S}^{d-1}$ such that $P_0(A_\nu) = 1$ while $P_0(A_l) = 0$ for $l \neq \nu$; (3) there exists a $\nu \in \mathbb{S}^{d-1}$ such that $P_0(A_l) = 1$ for l such that $l \cdot \nu > 0$ while $P_0(A_l) = 0$ for l such that $l \cdot \nu \leq 0$. The following corollary, which can be deduced from Theorem 1.8, shows that knowing that there is an l^* such that $P_0(A_{l^*}) = 1$ and $P_0(A_l) > 0$ for all l in a neighborhood of l^*, determines the value of $P_0(A_l)$ for all directions l.

Corollary 1.9. *The following are equivalent:*

(a) *There exists $l^* \in \mathbb{R}^d$ and some neighborhood $\mathcal{U}(l^*)$ such that $P_0(A_{l^*}) = 1$ and $P_0(A_l) > 0$ for all $l \in \mathcal{U}(l^*)$.*

(b) *There exists $\nu \in \mathbb{R}^d$ such that $P_0(A_l) = 1$ for l such that $l \cdot \nu > 0$, while $P_0(A_l) = 0$ for l such that $l \cdot \nu \leq 0$.*

In particular, this shows that in Theorem 1.7, condition (a) can be replaced by the a priori weaker condition (a) of this corollary.

In the rest of this paper we prove Theorem 1.8 and Corollary 1.9. In Section 2 we prove some preliminary results needed for the proofs and in Section 3 we apply them to prove the theorem and the corollary.

2 Preliminary results

The implications $(d) \Rightarrow (a) \Rightarrow (b)$ of Theorem 1.8 are obvious, so here we introduce the renewal structure and prove some preliminary results needed to show that $(b) \Rightarrow (c) \Rightarrow (d)$. For $l \in \mathbb{R}^d$ set

$$D_l := \inf\{n \in \mathbb{N} : X_n \cdot l < X_0 \cdot l\}$$

and for $B \subset \mathbb{R}^d$ define the first-exit time

$$D_B := \inf\{n \in \mathbb{N} : X_n \notin B\};$$

as usual, we set $\inf \emptyset := \infty$. We also define for $l \in \mathbb{R}^d$ and $s \in [0, \infty)$,

$$T_s^l := \inf\{n \in \mathbb{N} : X_n \cdot l > s\}.$$

Due to their linear independence, the vectors $l_1, \ldots l_d$ of Theorem 1.8 (b) give rise to the following 2^d cones:

$$C_\sigma := \cap_{k=1}^d \{x \in \mathbb{R}^d : \sigma_k(l_k \cdot x) \geq 0\}, \qquad \sigma \in \{-1, 1\}^d.$$

Furthermore, for $\lambda \in (0, 1]$ and $l \in \mathbb{R}^d \backslash \{0\}$ we will employ the notation

$$C_\sigma(\lambda, l) := \cap_{k=1}^d \{x \in \mathbb{R}^d : (\lambda \sigma_k l_k + (1 - \lambda) l) \cdot x \geq 0\}, \tag{2.1}$$

where the vectors defining the cone are now interpolations of the $\sigma_k l_k$ with l. Note that $C_\sigma(\lambda, l)$ is a non-degenerate cone with base of finite area if and only if the vectors $\lambda \sigma_k l_k + (1 - \lambda) l$, $k = 1, \ldots, d$, are linearly independent. In particular, $C_\sigma(1, l) = C_\sigma$ for all $\sigma \in \{-1, 1\}^d$ and l.

We will often choose σ such that $P_0(\cap_{k=1}^d A_{\sigma_k l_k}) > 0$, which under (1.5) is possible since we then have

$$1 = P_0(\cap_{k=1}^d A_{l_k} \cup A_{l_{-k}}) = P_0(\cup_\sigma \cap_{k=1}^d A_{\sigma_k l_k}) = \sum_\sigma P_0(\cap_{k=1}^d A_{\sigma_k l_k}). \tag{2.2}$$

For a given $\sigma \in \{-1, 1\}^d$ which will usually be clear from the context, we will frequently consider vectors $l \in \mathbb{R}^d$ satisfying the condition

$$\inf_{x \in C_\sigma \cap \mathbb{S}^{d-1}} l \cdot x > 0. \tag{2.3}$$

Note here that for σ such that $P_0(\cap_{k=1}^d A_{\sigma_k l_k}) > 0$ and l satisfying (2.3), the inequality $P_0(A_l) \geq P_0(\cap_{k=1}^d A_{\sigma_k l_k})$ implies that the measure $P_0(\cdot | A_l)$ is well-defined. For such l we will then show the existence of a $P_0(\cdot | A_l)$-a.s. asymptotic direction. The strategy of our proof is based to a significant part on that of Theorem 1.7.

We start with the following lemma which ensures that if with positive probability the random walk finally ends up in a cone, then the probability that it does so and never exits a half-space containing this cone is positive as well.

Lemma 2.1. *Let $\sigma \in \{-1, 1\}^d$ and $l \in \mathbb{R}^d$ such that (2.3) holds. Then*

$$P_0(\cap_{k=1}^d A_{\sigma_k l_k}) > 0 \Longrightarrow P_0(\cap_{k=1}^d A_{\sigma_k l_k} \cap \{D_l = \infty\}) > 0.$$

Proof. Assume $P_0(\cap_{k=1}^d A_{\sigma_k l_k} \cap \{D_l = \infty\}) = 0$. Then $\mathbb{P}$-a.s.

$$P_{0,\omega}(\cap_{k=1}^d A_{\sigma_k l_k} \cap \{D_l = \infty\}) = 0. \tag{2.4}$$

For $y \in \mathbb{R}^d$ with $l \cdot y \geq 0$ this implies

$$P_y(\cap_{k=1}^d A_{\sigma_k l_k} \cap \{D_{\{x:l\cdot x \geq 0\}} = \infty\}) = 0. \tag{2.5}$$

Indeed, if there existed such y with $\mathbb{P}(\{\omega \in \Omega | P_{y,\omega}(\cap_{k=1}^d A_{\sigma_k l_k} \cap \{D_{\{x:l\cdot x\geq 0\}} = \infty\}) > 0\}) > 0$ then for ω such that $P_{y,\omega}(\cap_{k=1}^d A_{\sigma_k l_k} \cap \{D_{\{x:l\cdot x\geq 0\}} = \infty\}) > 0$, a random walker starting in 0 would, with positive probability with respect to $P_{0,\omega}$, hit y before hitting $\{x : l \cdot x < 0\}$ (due to ellipticity) and from there on finally end up in C_σ without hitting $\{x : l \cdot x < 0\}$; this is a contradiction to (2.4), hence (2.5) holds.

Choosing a sequence $(y_n) \subset C_\sigma$ such that $l \cdot y_n \to \infty$ as $n \to \infty$ we therefore get

$$\begin{aligned} 0 &= P_{y_n}(\cap_{k=1}^d A_{\sigma_k l_k} \cap \{D_{\{x:l\cdot x\geq 0\}} = \infty\}) \\ &\geq P_0(\cap_{k=1}^d A_{\sigma_k l_k} \cap \{D_{\{x:l\cdot x\geq -l\cdot y_n\}} = \infty\}) \to P_0(\cap_{k=1}^d A_{\sigma_k l_k}) \end{aligned}$$

as $n \to \infty$. To obtain the inequality we employed the translation invariance of $\mathbb{P}$ as well as the monotonicity of events. □

The following lemma will be employed to set up a renewal structure; it can in some way be seen as an analog to Lemma 1 of [Sim07].

Lemma 2.2. *Let $\sigma \in \{-1, 1\}^d$ such that $P_0(\cap_{k=1}^d A_{\sigma_k l_k}) > 0$. Then for each l such that (2.3) holds, one has*

$$P_0(\{D_{C_\sigma(\lambda,l)} = \infty\}) > 0 \tag{2.6}$$

for $\lambda > 0$ small enough.

Proof. Lemma 2.1 implies $P_0(\cap_{k=1}^d A_{\sigma_k l_k} \cap \{D_l = \infty\}) > 0$. Due to the ellipticity of the walk and the independence of the environment we therefore obtain

$$P_0\Big(\{X_1 \cdot l > 0\} \bigcap \cap_{k=1}^d A_{\sigma_k l_k}(X_{1+\cdot} - X_1) \cap \{D_l(X_{1+\cdot} - X_1) = \infty\}\Big) > 0, \tag{2.7}$$

where we name explicitly the path $X_{1+\cdot} - X_1$ to which the corresponding events $A_{\sigma_k l_k}$ and D_l refer. Each path of the event in (2.7) is fully contained in $C_\sigma(\lambda, l)$ for $\lambda > 0$ small enough. Thus, the continuity from above of P_0 yields

$$P_0\Big(\{D_{C_\sigma(\lambda,l)} = \infty\} \cap \{X_1 \cdot l > 0\} \bigcap \cap_{k=1}^d A_{\sigma_k l_k}(X_{1+\cdot} - X_1) \cap \{D_l(X_{1+\cdot} - X_1) = \infty\}\Big) > 0 \tag{2.8}$$

for all $\lambda > 0$ small enough. □

Employing Lemma 2.2, for $\sigma \in \{-1, 1\}^d$ with $P_0(\cap_{k=1}^d A_{\sigma_k l_k}) > 0$ as in [Sim07] we can introduce a cone renewal structure, where we choose $l \in \mathbb{R}^d$ such that (2.3) is fulfilled and the cone to work with is $C_l := C_\sigma(\lambda, l)$, where we fixed $\lambda > 0$ small enough as in the statement of Lemma 2.2. Note that for fixed l the set $C_\sigma(\lambda, l)$ is indeed a cone as long as $\lambda > 0$ is chosen small enough (since the defining vectors in (2.1) are linearly independent). We define

$$S_0^l := T_0^l, \quad R_0^l := D_{X_{S_0^l} + C_l} \circ \theta_{S_0^l} + S_0^l, \quad M_0^l := \max\{X_n \cdot l : 0 \leq n \leq R_0^l\}$$

and inductively for $k \geq 1$:

$$S_k^l := T_{M_{k-1}^l}^l, \quad R_k^l := D_{X_{S_k^l} + C_l} \circ \theta_{S_{k-1}^l} + S_k^l, \quad M_k^l := \max\{X_n \cdot l : 0 \leq n \leq R_k^l\},$$

where for $x \in \mathbb{Z}^d$ by $x + C_l$ we denote the cone C_l shifted such that its apex lies at x. Furthermore, set

$$K^l := \inf\{k \in \mathbb{N} : S_k^l < \infty, R_k^l = \infty\}$$

as well as

$$\tau_1^l := S_{K^l}^l,$$

i.e. τ_1^l is the first time at which the walk reaches a new maximum in direction l and never exits the cone C_l shifted to $X_{\tau_1^l}$. We define inductively the sequence of cone renewal times with respect to C_l by

$$\tau_k^l := \tau_1^l(X_{\cdot+\tau_{k-1}^l} - X_{\tau_{k-1}^l}) + \tau_{k-1}^l$$

for $k \geq 2$.

The following lemma shows that under the conditions of Lemma 2.2 the sequence τ_k^l is well-defined on A_l. It can be seen as an analog to Proposition 2 of [Sim07].

Lemma 2.3. *Let $\sigma \in \{-1,1\}^d$ such that $P_0(\cap_{k=1}^d A_{\sigma_k l_k}) > 0$ and choose l and λ such that (2.3) and (2.6) hold. Then $P_0(\cdot|A_l)$-a.s. one has $K^l < \infty$.*

Proof. Employing Lemma 2.2, the proof takes advantage of standard renewal arguments and is analogous to the proof of Proposition 2 in [Sim07] or Proposition 1.2 in [SZ99]. □

Lemma 2.4. *Let $\sigma \in \{-1,1\}^d$ such that $P_0(\cap_{k=1}^d A_{\sigma_k l_k}) > 0$ and choose l and λ such that (2.3) and (2.6) hold. Then $((X_{\tau_1^l \wedge \cdot}, \tau_1^l), \dots, (X_{(\tau_k^l+\cdot)\wedge\tau_{k+1}^l} - X_{\tau_k^l}, \tau_{k+1}^l - \tau_k^l)), \dots$ are independent under $P_0(\cdot|A_l)$ and for $k \geq 1$, $((X_{(\tau_k^l+\cdot)\wedge\tau_{k+1}^l} - X_{\tau_k}^l), \tau_{k+1}^l - \tau_k^l)$ under $P_0(\cdot|A_l)$ is distributed like $(X_{\tau_1^l\wedge\cdot}, \tau_1^l)$ under $P_0(\cdot|\{D_{C_l} = \infty\})$.*

Proof. The proof is analogous to the proof of Corollary 1.5 in [SZ99]. □

The following lemma has been derived in Simenhaus' thesis [Sim08] (Lemma 2 in there). Here we state it and prove it under a slightly weaker assumption.

Lemma 2.5. *Let $\sigma \in \{-1,1\}^d$ such that $P_0(\cap_{k=1}^d A_{\sigma_k l_k}) > 0$ and choose $l \in \mathbb{Z}^d$ and λ such that (2.3) and (2.6) hold and the g.c.d. of the coordinates of l is 1. Then*

$$E_0(X_{\tau_1^l} \cdot l|\{D_{C_l} = \infty\}) = \frac{1}{P_0(\{D_{C_l} = \infty\}|A_l) \lim_{i\to\infty} P_0(\{T_{i-1}^l < \infty, X_{T_{i-1}^l} \cdot l = i\})} < \infty$$

and

$$E_0(X_{\tau_1^l}|\{D_{C_l} = \infty\}) \tag{2.9}$$

is well-defined.

Remark 2.6. A fundamental consequence of working with the cone renewal structure instead of working with slabs is the existence of (2.9), see Proposition (2.7) also.

Proof. The proof leans on the proof of Lemma 3.2.5 in [TZ04] which is due to Zerner. Due to the strong Markov property and the independence and translation invariance of the environment we have for $i > 0$:

$$\begin{aligned} P_0(\{\exists k \geq 1 : X_{\tau_k^l} \cdot l = i\} \cap A_l) &= \sum_{x\in\mathbb{Z}^d, l\cdot x = i} \mathbb{E}P_{0,\omega}(\{T_{i-1}^l < \infty, X_{T_{i-1}^l} = x, D_{C_l + X_{T_{i-1}^l}} \circ \theta_{T_{i-1}^l} = \infty\}) \\ &= \sum_{x\in\mathbb{Z}^d, l\cdot x = i} \mathbb{E}P_{0,\omega}(\{T_{i-1}^l < \infty, X_{T_{i-1}^l} = x\}) P_{x,\omega}(\{D_{C_l + x} = \infty\}) \\ &= P_0(\{T_{i-1}^l < \infty, X_{T_{i-1}^l} \cdot l = i\}) P_0(\{D_{C_l} = \infty\}). \end{aligned} \tag{2.10}$$

At the same time using $\{\tau_1^l < \infty\} = A_l$, a fact which is proven similarly to Proposition 1.2 of [SZ99], we compute

$$\begin{aligned} &\lim_{i\to\infty} P_0(\{\exists k \geq 1 : X_{\tau_k^l} \cdot l = i\}|A_l) \\ &\quad = \lim_{i\to\infty} P_0(\{\exists k \geq 2 : X_{\tau_k^l} \cdot l = i\}|A_l) \\ &\quad = \lim_{i\to\infty} \sum_{n\geq 1} P_0(\{\exists k \geq 2 : X_{\tau_k^l} \cdot l = i\} \cap \{X_{\tau_1^l} \cdot l = n\}|A_l) \\ &\quad = \lim_{i\to\infty} \sum_{n\geq 1} P_0(\{\exists k \geq 2 : (X_{\tau_k^l} - X_{\tau_1^l}) \cdot l = i - n\} \cap \{X_{\tau_1^l} \cdot l = n\}|A_l) \\ &\quad = \lim_{i\to\infty} \sum_{n\geq 1} P_0(\{\exists k \geq 2 : (X_{\tau_k^l} - X_{\tau_1^l}) \cdot l = i - n\}|A_l) P_0(\{X_{\tau_1^l} \cdot l = n\}|A_l), \end{aligned} \tag{2.11}$$

where to obtain the last equality we took advantage of Lemma 2.4. Blackwell's renewal theorem in combination with Lemma 2.4 now yields

$$\lim_{i\to\infty} P_0(\{\exists k \geq 2 : (X_{\tau_k^l} - X_{\tau_1^l}) \cdot l = i - n\}|A_l) = \frac{1}{E_0(X_{\tau_1^l} \cdot l|\{D_{C_l} = \infty\})}$$

and thus (2.11) implies

$$\lim_{i\to\infty} P_0(\{\exists k \geq 1 : X_{\tau_k^l} \cdot l = i\}|A_l) = \frac{1}{E_0(X_{\tau_1^l} \cdot l|\{D_{C_l} = \infty\})}.$$

Therefore, taking into consideration (2.10) we infer

$$E_0(X_{\tau_1^l} \cdot l|\{D_{C_l} = \infty\}) = \frac{1}{P_0(\{D_{C_l} = \infty\}|A_l) \lim_{i\to\infty} P_0(\{T_{i-1}^l < \infty, X_{T_{i-1}^l} \cdot l = i\})}. \tag{2.12}$$

It remains to show that the right hand side of (2.12) is finite. Writing $l_{max} := \max\{|l_1|, \dots, |l_d|\}$ for the maximum of the absolute values of the coordinates of l we have

$$\sum_{i=k}^{k+l_{max}-1} P_0(\{T_{i-1}^l < \infty, X_{T_{i-1}^l} \cdot l = i\}) \geq \sum_{i=k}^{k+l_{max}-1} P_0(\{T_{i-1}^l < \infty, X_{T_{k-1}^l} \cdot l = i\}) \geq P_0(A_l), \quad \forall k \in \mathbb{N},$$

where the first inequality follows since $\{X_{T_{k-1}^l} \cdot l = i\} \subseteq \{X_{T_{i-1}^l} \cdot l = i\}$ for all $k \in \mathbb{N}$ and $i \in \{k, \dots, k + l_{max} - 1\}$. This now yields $\lim_{i\to\infty} P_0(\{T_{i-1}^l < \infty, X_{T_{i-1}^l} \cdot l = i\}) \geq l_{max}^{-1} P_0(A_l) > 0$, whence due to (2.12) we obtain

$$E_0(X_{\tau_1^l} \cdot l|\{D_{C_l} = \infty\}) < \infty. \tag{2.13}$$

Since on $\{D_{C_l} = \infty\}$ there exists a constant $C > 0$ such that $|X_{\tau_1^l}| \leq C X_{\tau_1^l} \cdot l$, we infer as a direct consequence of (2.13) that (2.9) is well-defined. □

We can now employ the above renewal structure to obtain an a.s. constant asymptotic direction on A_l.

Proposition 2.7. *Let $\sigma \in \{-1, 1\}^d$ such that $P_0(\cap_{k=1}^d A_{\sigma_k l_k}) > 0$ and choose $l \in \mathbb{Z}^d$ and λ such that (2.3) and (2.6) hold and the g.c.d of the coordinates of l is 1. Then $P_0(\cdot|A_l)$-a.s.*

$$\lim_{n\to\infty} \frac{X_n}{|X_n|} = \frac{E_0(X_{\tau_1^l}|\{D_{C_l} = \infty\})}{|E_0(X_{\tau_1^l}|\{D_{C_l} = \infty\})|}.$$

Remark 2.8. In particular, this proposition implies that the limit does not depend on the particular choice of l nor λ (for λ sufficiently small). Note that the independence of l stems from the fact that if l_1, l_2 satisfy (2.3) we have $P_0(A_{l_1} \cap A_{l_2}) > 0$.

Proof. Due to Lemmas 2.2 to 2.5 we may apply the law of large numbers to the sequence $(X_{\tau_k^l})_{k\in\mathbb{N}}$ yielding

$$\frac{X_{\tau_k^l}}{k} \to E_0(X_{\tau_1^l}|\{D_{C_l} = \infty\}) \quad P_0(\cdot|A_l) - a.s., \quad k \to \infty,$$

and hence

$$\frac{X_{\tau_k^l}}{|X_{\tau_k^l}|} \to \frac{E_0(X_{\tau_1^l}|\{D_{C_l} = \infty\})}{|E_0(X_{\tau_1^l}|\{D_{C_l} = \infty\})|} \quad P_0(\cdot|A_l) - a.s., \quad k \to \infty.$$

Using standard methods to estimate the intermediate terms (cf. p. 9 in [Sim07]) one obtains

$$\lim_{n\to\infty} \frac{X_n}{|X_n|} = \frac{E_0(X_{\tau_1^l}|\{D_{C_l} = \infty\})}{|E_0(X_{\tau_1^l}|\{D_{C_l} = \infty\})|} \quad P_0(\cdot|A_l) - a.s.$$

□

The following two results will be needed to obtain results about transience in directions orthogonal to the asymptotic direction.

Lemma 2.9. *Let $(Y_n)_{n\in\mathbb{N}}$ be an i.i.d. sequence on some probability space $(\mathcal{X},\mathcal{F},P)$ with expectation $EY_1=0$ and variance $EY_1^2\in(0,\infty]$. Then, for $S_n:=\sum_{k=1}^n Y_k$ we have $P(\{\liminf_{n\to\infty} S_n=-\infty\})=P(\{\limsup_{n\to\infty} S_n=\infty\})=1$.*

Proof. We only prove $P(\{\liminf_{n\to\infty} S_n=-\infty\})=1$, the remaining equality is proved in an analog way. Setting $\varepsilon:=(-\operatorname{ess\,inf} Y_1/2)\wedge 1$ one can show for all $x\in\mathbb{R}$, using the strong Markov property at the entrance times of S_n to the interval $[x,x+\varepsilon]$, that $P(\{\liminf_{n\to\infty} S_n\in[x,x+\varepsilon]\})=0$. This then implies $P(\{\liminf_{n\to\infty} S_n=\pm\infty\})=1$. But Kesten's result in [Kes75] yields $\liminf_{n\to\infty} S_n/n>0$ $P(\cdot\cap\{\liminf_{n\to\infty} S_n=\infty\})$-a.s., while by the strong law of large numbers we have $\lim_{n\to\infty} S_n/n=0$ P-a.s. This yields $P(\{\liminf_{n\to\infty} S_n=\infty\})=0$ and hence finishes the proof. $\square$

Lemma 2.10. *Let $l\in\mathbb{R}^d$ such that*

$$P_0(\{\lim_{n\to\infty} X_n/|X_n|=l\})>0. \tag{2.14}$$

Then, for $l^*\in\mathbb{R}^d$ such that $l^*\cdot l=0$ one has $P_0((A_{l^*}\cup A_{-l^*})\cap A_l)=0$.

Proof. We choose a basis $l_1,\dots,l_d$ of $\mathbb{R}^d$ and σ such that l is contained in the interior of the cone C_σ corresponding to $l_1,\dots,l_d$ and (2.3) is satisfied. Furthermore, by (2.14) and Lemma 2.2 we may choose λ such that condition (2.6) is satisfied for the corresponding cone $C_\sigma(\lambda,l)$. Lemma 2.3 yields that the sequence $(\tau_k^l)_{k\in\mathbb{N}}$ is well defined and Lemmas 2.4 and 2.5 yield that under $P_0(\cdot|A_l)$ the sequence $((X_{\tau_2^l}-X_{\tau_1^l})\cdot l^*,(X_{\tau_3^l}-X_{\tau_2^l})\cdot l^*,\dots)$ is i.i.d. with expectation 0, the latter being due to the validity of Lemma 2.5 as well as (1.6) and $l^*\cdot l=0$. Indeed, Proposition 2.7 yields

$$E_0(X_{\tau_1^l}\cdot l^*|\{D_{C_\sigma(\lambda,l)}=\infty\})=|E_0(X_{\tau_1^l}|\{D_{C_\sigma(\lambda,l)}=\infty\})|\underbrace{\lim_{k\to\infty}\frac{X_{\tau_k^l}}{|X_{\tau_k^l}|}}_{=l}\cdot l^*=0\quad P_0(\cdot|A_l)-a.s.$$

Applying Lemma 2.9 to the sequence $((X_{\tau_2^l}-X_{\tau_1^l})\cdot l^*,(X_{\tau_3^l}-X_{\tau_2^l})\cdot l^*,\dots)$ yields $P_0((A_{l^*}\cup A_{-l^*})\cap A_l)=0$. $\square$

3 Proof of Theorem 1.8 and Corollary 1.9

3.1 Proof of Theorem 1.8

We first prove that condition (b) implies (c). Note that due to Lemma 2.10 and (2.3), we obtain $P_0(\{\lim_{n\to\infty} X_n/|X_n|\in\cup_\sigma\partial C_\sigma\})=0$. We now choose σ such that $P_0(\cap_{k=1}^d A_{\sigma_k l_k})>0$ and observe $\cup_l\{x\in\mathbb{R}^d: l\cdot x>0\}=\operatorname{int}\cup_{\sigma^*\neq-\sigma} C_{\sigma^*}$; here, with "int" we denote the interior of a set and the union is taken over all vectors $l\in\mathbb{Z}^d$ that satisfy (2.3) and for which the g.c.d. of the coordinates of l is 1. Hence, letting l vary over all such vectors, Proposition 2.7 yields $P_0(\cdot|\cup_{\sigma^*\neq-\sigma}\cap_{k=1}^d A_{\sigma_k^* l_k})$-a.s.

$$\lim_{n\to\infty}\frac{X_n}{|X_n|}=\frac{E_0(X_{\tau_1^l}|\{D_{C_l}=\infty\})}{|E_0(X_{\tau_1^l}|\{D_{C_l}=\infty\})|}=:\nu, \tag{3.1}$$

which due to Remark 2.8 is independent of the respective l chosen. Now if $P_0(\cup_{\sigma^*\neq-\sigma}\cap_{k=1}^d A_{\sigma_k^* l_k})=1$ this finishes the proof and the result is equivalent to Theorem 1.7 obtained in [Sim07]. Thus, assume

$$P_0(\cup_{\sigma^*\neq-\sigma}\cap_{k=1}^d A_{\sigma_k^* l_k})\in(0,1). \tag{3.2}$$

In the same manner as before we obtain for any $l'\in\mathbb{Z}^d$ with coordinates of g.c.d. 1 and satisfying (2.3) with σ replaced by $-\sigma$

$$\lim_{n\to\infty}\frac{X_n}{|X_n|}=\frac{E_0(X_{\tau_1^{l'}}|\{D_{C_{l'}}=\infty\})}{|E_0(X_{\tau_1^{l'}}|\{D_{C_{l'}}=\infty\})|} \tag{3.3}$$

$P_0(\cdot|\cap_{k=1}^d A_{-\sigma_k l_k})$-a.s. with hopefully self-explaining notations. Now Proposition 1 of [Sim07] states that if two elements $\nu\neq\nu'$ of $\mathbb{S}^{d-1}$ occur with positive probability each with respect to P_0 as asymptotic directions, then $\nu=-\nu'$. Thus, (3.1) to (3.3) imply that the limit in (3.3) equals $-\nu$, and (3.3) holds $P_0(\cdot|A_{-\nu})$-a.s. This yields (c).

Now with respect to the implication $(c)\Rightarrow(d)$ note that the only thing that is not obvious at a first glance is that $l\cdot\nu=0$ implies $P_0(A_l\cup A_{-l})=0$. However, Lemma 2.10 yields $P_0((A_l\cup A_{-l})\cap(A_\nu\cup A_{-\nu}))=0$ which due to $P_0(A_\nu\cup A_{-\nu})=1$ yields the desired result.

3.2 Proof of Corollary 1.9

We only have to prove $(a) \Rightarrow (b)$. Given (a), Theorem 1.8 yields the existence of $\nu \in \mathbb{S}^{d-1}$ such that

$$P_0(A_\nu \cup A_{-\nu}) = 1 \tag{3.4}$$

and (1.6) holds.

Now if $l^* \cdot \nu \neq 0$ then $P_0(A_\nu \cap A_{l^*}) = 1$ or $P_0(A_{-\nu} \cap A_{l^*}) = 1$, respectively, and hence $P_0(A_\nu) = 1$ or $P_0(A_{-\nu}) = 1$, which due to Theorem 1.7 finishes the proof. Thus, assume

$$l^* \cdot \nu = 0 \tag{3.5}$$

from now on. Then Lemma 2.10 yields $P_0((A_{l^*} \cup A_{-l^*}) \cap (A_\nu \cup A_{-\nu})) = 0$ which due to (3.4) implies $P_0(A_{l^*} \cup A_{-l^*}) = 0$, a contradiction to assumption (a).

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